A NOTE ON GEODESIC CIRCLES.

By J. K. Whittemore.

Bianchi defines a geodesic circle as the locus of a point on a surface at a constant geodesic distance from a fixed point of the surface.* In this note I use the term in this sense. Darboux calls a curve of constant geodesic curvature a geodesic circle.† Bianchi states in a footnote‡ that all the geodesic circles of a surface are curves of constant geodesic curvature, when the total curvature of the surface is constant.

In this note I shall prove a set of more general theorems from which Bianchi's statement and its converse follow at once. I use the term "geodesic circle" in Bianchi's sense. My theorems are the following:

Theorem 1. *If, on a surface, there exists a family of concentric geodesic circles such that the geodesic curvature of each curve of the family is constant, then the total curvature of the surface is constant along each curve of the family, and the surface is applicable to a surface of revolution so that the geodesic circles fall on the circles of latitude of this surface.*

Theorem 2. *Conversely, if, on a surface, there exists a family of concentric geodesic circles such that the total curvature of the surface is constant along each curve of the family, then the geodesic curvature of each geodesic circle is constant, and the surface is applicable to a surface of revolution by the former theorem.*

Theorem 3. *Finally, if a surface is applicable to a surface of revolution so that the members of a family of concentric geodesic circles of the surface fall upon the circles of latitude of the surface, then the geodesic circles are curves of constant geodesic curvature and of constant total curvature.*

I suppose the common centre of the geodesic circles to be an ordinary point of the surface. I choose as curvilinear coordinates on the surface, the geodesic distance, u, from the common centre of the circles, and the angle, v, which a geodesic makes at this point with some fixed direction on the surface at this point. Then the linear element of the surface has the form,

$$ds^2 = du^2 + C^2\,dv^2,$$

* Bianchi. *Vorlesungen über Differentialgeometrie*, p. 160.

† Darboux. *Théorie générale des surfaces*, vol. 3, p. 151.

‡ *l. c.*, p. 162.

where $$C_{\substack{u=0\\v=v}} = 0, \qquad \left(\frac{\partial C}{\partial u}\right)_{\substack{u=0\\v=v}} = 1.$$

The geodesic curvature of one of the geodesic circles is given by the formula:

$$\frac{1}{\rho_u} = -\frac{1}{C}\frac{\partial C}{\partial u}.$$

The total curvature of the surface at any point is*

$$\kappa = -\frac{1}{C}\frac{\partial^2 C}{\partial u^2}$$

Consider first Theorem 3. The element of arc dS of a surface of revolution can be expressed in the form

$$dS^2 = d\sigma^2 + \phi(\sigma)^2 d\theta^2,$$

where $d\sigma$ denotes the element of arc of a meridian curve, $y = f(x)$, and $\phi(\sigma) = y$; the longitude being measured by θ. According to the hypothesis of the theorem, it is possible to determine u and v as functions of σ and θ such that, when $u =$ const., $\sigma =$ const. also and furthermore that $dS^2 = ds^2$, or

$$du^2 + C^2 dv^2 = d\sigma^2 + \phi^2 d\theta^2. \tag{1}$$

Let $u = u(\sigma, \theta)$, $v = v(\sigma, \theta)$. Then, from the first condition it follows that

$$\frac{\partial u}{\partial \theta} = 0,$$

Substituting for du and dv their values in terms of $d\sigma$ and $d\theta$, we show further that

$$\frac{\partial u}{\partial \sigma} = 1, \qquad \frac{\partial v}{\partial \sigma} = 0.$$

Hence $u = \sigma$ and v is a function of θ alone. Since the assignment of the individual geodesics $v = v_1$ to the individual meridians $\theta = \theta_1$ is still arbitrary, we may make this assignment in such a way that $v = \theta$. It then follows from (1) that

$$C^2 = \phi(\sigma)^2,$$

i. e., that C is a function of u alone. The same will be true of $1/\rho_u$ and κ; hence the theorem.

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* For these formulas see Bianchi, *l. c.* pp. 161, 148, 159.

Let us now suppose, to prove Theorem 1, that the geodesic circles, $u =$ constant, are curves of constant geodesic curvature. Then

$$\frac{1}{\rho_u} = -\frac{1}{C}\frac{\partial C}{\partial u} = -U'$$

where U is a function of u alone. We have by integration

$$C = e^{U+V}$$

where V is a function of v alone.

Now, since $\left(\frac{\partial C}{\partial u}\right)_{u=0} = 1$, it follows that

$$1 = U'(0)e^{U(0)+V(v)},$$

and hence that V is a constant. C is, therefore, a function of u alone, and Theorem 1 is proved.*

Finally, to prove Theorem 2, suppose that the geodesic circles, $u =$ constant, are for the surface curves of constant total curvature. Then is

$$\kappa = -\frac{1}{C}\frac{\partial^2 C}{\partial u^2} = -F(u) \tag{2}$$

where $F(0)$ is finite, since the centre of the geodesic circles is an ordinary point of the surface. In order to integrate equation (2), let U be a solution of the ordinary differential equation

$$U'' + U'^2 = F(u)$$

which vanishes for $u = 0$. Then we have

$$\frac{1}{C}\frac{\partial^2 C}{\partial u^2} = U'' + U'^2$$

Next, replace C by z where

$$C = z\,e^U;$$

z is an unknown function of u and v, which must however vanish for $u = 0$. For the determination of z we have the equation:

$$\frac{\partial^2 z}{\partial u^2} + 2U'\frac{\partial z}{\partial u} = 0.$$

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* For a proof of the theorem that, if the element of arc of a surface can be written in the form $dS^2 = Edu^2 + Gdv^2$, where E, G are functions of u alone, the surface is applicable to a surface of revolution, *cf.* Picard, *Traité d'analyse*, vol. 1, p. 423.

Integrating this equation, the general value of z is found to be

$$z = e^V \int_0^u e^{-2U}\, du + V_1,$$

where V and V_1 are arbitrary functions of v. Since now z and u vanish together, V_1 is identically zero, and

$$C = e^{U+V} \int_0^u e^{-2U}\, du.$$

Furthermore, since for $u = 0$, we have

$$\frac{\partial C}{\partial u} = 1, \qquad U = 0,$$

it follows that for all values of v

$$e^V = 1.$$

Hence V is identically zero, and

$$C = e^u \int_0^u e^{-2U}\, du$$

and is a function of u alone. Theorem 2 follows at once.*

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* Picard, *l. c.*